SIMPLIFIED ANALYSIS OF A STOCHASTIC SYSTEM

Vladimír HERLES

Institute of Information Theory and Automation, Czechoslovak Academy of Sciences, 182 08 Prague 8

Received November 3rd, 1977

Contradictious results published by different authors about the dynamics of systems with random parameters have been examined. Statistical analysis of the simple 1st order system proves that the random parameter can cause a systematic difference in the dynamic behaviour that cannot be (in general) described by the usual constant-parameter model with the additive noise at the output.

The majority of existing methods for a modelling of chemical objects and processes[1-3] is mostly based on the following scheme: *a*) statement of a general mathematical model; *b*) experimental identification of its coefficients; *c*) verification of the model on the reality.

As a first step, the physico-chemical analysis is commonly used for the determination of the general mathematical description of the system (deterministic model). The aim of the second and the third step is an evaluation (identification) of unknown parameters and verification of the whole model in real or simulated conditions. Quality of the model can be judged either using some explicitely defined criterion or at least subjectively (visual coincidence of the response *etc.*).

The just outlined scheme, more or less common for all natural sciences, has some specific features in chemical applications. At first, the exact analytical formulation of the mathematical model is often considerably complicated and laborous (distributed parameters, nonlinearities *etc.*) and the simplification of it is not always straightforward. Even the experimental verification of the finded model is not without problems either. High level of noise, different quality, accuracy and reliability of experimental equipment effect directly the achievable precision of results.

Extremely high level of random disturbances calls for statistical processing either in data processing or in the statement of the stochastic mathematical model that could be more suitable for the description of the real system than the deterministic one.

Modern theory of automatic control often deals with the term stochastic system which can be described mostly as a block scheme in Fig. 1. Object S is described as a model with constant or slowly changing coefficients. The input $x(t)$ may by either random or deterministic one. On the output we can observe only a mixture of an ideal output and an additive random disturbance. It does not matter whether the object is relatively simple or complex one (several inputs and outputs, feedbacks *etc.*); the main idea of this approach consists in the constant-coefficient model and in the concentration of all disturbances in one point of the signal way. For linear systems it is not important whether this point lies at the output, input or at some other place of the signal way, because the influence can be simply transformed anywhere else.

The described approach has been largely analysed and frequently used in many applications. Despite of its simplicity it describes quite satisfactorily numerous technical and scientific problems. One of its main advantage consists in the possibility of diminishing the disturbing influence of the random component by averaging (*e.g.* repetitive measurements, correlation technique *etc.*).

Although this fact facilitates and even enables the solution of many problems, it cannot be used in every case. As an example it may be mentioned a two-phase continuous stirred tank reactor with randomly fluctuating instantaneous level (volume) of an inner liquid. If the random component is significant with respect to the mean value of the volume, the mathematical description yields to the model with variable coefficients (time constants) for which the model mentioned above (Fig. 1.) is substantionally non-adequate.

Systems with randomly varying parameters and their dynamic properties have been examined by several authors either in the theoretical or experimental field. Interesting results have been published by King[1,2], who theoretically showed, that even a 1^{st} order system with variable time constant can occur substantionally different dynamic properties than a corresponding constant-parameter system. In the mentioned example the random parameter causes an increasing of an effective (mean) time constant of the system. On the other hand some other authors like Berrymann and Himmelblau[3] in a simulated experimental analysis proved that no significant differences occurred between the dynamics of the constant and random parameter systems (in the mean sense).

With respect to this discrepancy and having in mind that the King's conclusions particularly could call for the principal revision of the widely used identification methods, experimental verification of theoretical results *etc.*, there is analysed a simple example for judging the significance of mentioned contradictious results.

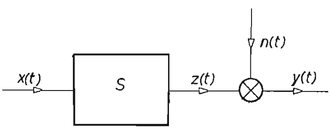

FIG. 1
Stochastic Model of the System with Constant Parameters

THEORETICAL

STATEMENT OF THE PROBLEM

Let us have the simplest dynamic 1st order system described by an ordinary differential equation with time-variable coefficients:

$$y' + ay = bx\,. \tag{1}$$

Dynamic member of this type – either with constant or variable coefficients – represents a basic element for the description of the dynamics of almost any chemical equipment. Accordingly, the deviation detected in this case will necessarily influence the dynamics of all other more complicated equipments.

Equation (*1*) with time-constant coefficients has the solution (weighting function – response to an input Dirac impulse) in the form (Fig. 2):

$$y(t) = b \exp\left(-at\right). \tag{2}$$

Consider now that the coefficients in (*1*) randomly vary in time. Such equation describes *e.g.* the dynamics of a continuous reactor with fluctuating volume flow rate of the reacting mixture. Let us take following assumptions:

a) The system response is a continuous function of time for any $t \neq 0$.
b) The changes of parameters can occur only at discrete time instants $(t_i : i = 0, 1, 2 \ldots;\ t_0 = 0)$. (*3*)
c) During the interval $t_i \leqq t < t_{i+1}$ the parameters remain unchanged.

The first assumption is for real (inertial) systems fulfilled automatically. The others can be approximated with a sufficient accuracy by a suitable choice of instants t_i.

The first-order stochastic system described by Eq. (*1*) will be analysed for the case

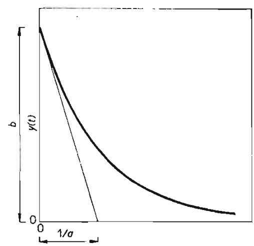

Fig. 2

Impulse Response of the System (*1*) with Constant Coefficients ($y_0 = 1$)

that the parameter a (inverse value of the time "constant" of the system) is randomly changed. In order to fulfill mentioned assumptions the values b must then be changed too – depending on a. Independent changes of both parameters (*e.g.* one of them kept constant) as it is sometimes assumed, entail a discontinuous system response contradicting the physically motivated assumption (*3a*).

The impulse response of such stochastic system is given as a sum of elementary responses $y_i(t)$ that form a solution of Eq. (*1*) within one time interval, where the vector of parameters does not change:

$$\left.\begin{aligned} y_i(t) &= y(t) \quad \text{for} \quad t_i \leqq t < t_{i+1} \\ y_i(t) &= 0 \qquad \text{for other} \quad t \end{aligned}\right\} \tag{4}$$

$$y(t) = \sum_{i=0}^{\infty} y_i(t)\,. \tag{5}$$

The elementary impulse response is then given (Fig. 3):

$$\left.\begin{aligned} y_i(t) &= y_i \exp\left[-a_i(t - t_i)\right] \quad \text{for} \quad t_i \leqq t < t_{i+1} \\ y_i(t) &= 0 \qquad\qquad\qquad\qquad \text{for other} \quad t\,, \end{aligned}\right\} \tag{6}$$

where

$$y_i = y_i(t_i)\,. \tag{7}$$

Due to assumption of continuity:

$$\lim_{t \to t_{i+1}^-} y_i(t) = y_{i+1}(t_{i+1}) = y_{i+1}\,. \tag{8}$$

Without loss of generality we may put $t_0 = 0$:

$$\begin{aligned} y_0(t_1) &= y_1 = y_0 \exp\left[-a_1 t_1\right] \\ y_1(t_2) &= y_2 = y_1 \exp\left[-a_2(t_2 - t_1)\right] \\ y_2(t_3) &= y_3 = y_2 \exp\left[-a_3(t_3 - t_2)\right] \\ &\;\vdots \\ y_{i-1}(t_i) &= y_i = y_{i-1} \exp\left[-a_i(t_i - t_{i-1})\right]. \\ &\;\vdots \end{aligned} \tag{9}$$

After sequential substitution we obtain:

$$y_i = y_0 \exp\left\{-\sum_{j=1}^{i} a_j(t_j - t_{j-1})\right\}. \tag{10}$$

For a special case of equidistant time instants

$$t_i - t_{i-1} = \Delta t = \text{const} \quad (11)$$

$$y_i = y_0 \exp \{-\Delta t \sum_{j=1}^{i} a_j\} . \quad (12)$$

Final impulse response is then given by the substitution of Eq. (*12*) or (*10*) in Eq. (*5*) (Fig. 3).

STATISTICAL CHARACTERISTICS OF THE RESPONSE

As it has been assumed, the values a_j depend on the external influences. They can be described as random variables with some statistical properties, and the whole response as a random process. Its value in every time instant is a random variable.

Let us assume that values a_j are mutually independent random variables, stationary in the wide sense with the mean $\bar{a}$, standart deviation σ_a and probability density $p(a_j)$.

The mean value of the response in the time instant t_i is given:

$$E\{y_i\} = E\{y_0 \exp [-\Delta t \sum_{j=1}^{i} a_j]\} =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} y_i p(a_1; a_2; \dots a_i) \, da_1 \, da_2 \dots da_i =$$

$$= y_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{j=1}^{i} [\exp(-a_j \Delta t) \, p(a_j)] \, da_1 \, da_2 \dots da_i =$$

$$= y_0 \prod_{j=1}^{i} \int_{-\infty}^{+\infty} \exp(-a_j \Delta t) \, p(a_j) \, da_j . \quad (13)$$

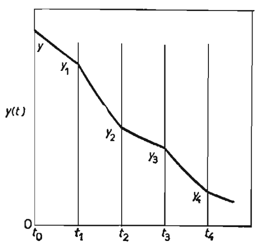

FIG. 3

Impulse Response of the Non-Stationary System

Similarly the variance:

$$D\{y_i\} = E\{[y_i - E\{y_i\}]^2\} = E\{y_i^2\} - E^2\{y_i\} =$$

$$= E\{y_0^2 \exp(-2\Delta t \sum_{j=1}^{i} a_j)\} - E^2\{y_i\} =$$

$$= y_0^2 \prod_{j=1}^{i} \int_{-\infty}^{+\infty} \exp(-2a_j \Delta t)\, p(a_j)\, da_j - E^2\{y_i\}\,. \tag{14}$$

The further procedure depends on the type of probability distribution of the random vector a_j. As an example, two typical distribution laws will be analysed and compared (other types of distributions can be used similarly):

Normal Distribution

The probability density of the parameter a_j has the form

$$p(a_j) = (1/\sigma_a \sqrt{(2\pi)}) \,.\, \exp[-(a_j - \bar{a})^2/2\sigma_a^2]\,, \tag{15}$$

where $\bar{a}$ and σ_a are the mean and standard deviation of the parameter a_j. Due to the assumption of stationarity, both of them are constant in time.

The mean value of the response *(13)* is then given:

$$E\{y_i\} = y_0 \prod_{j=1}^{i} \int_{-\infty}^{+\infty} \exp(-a_j \Delta t)\, p(a_j)\, da_j =$$

$$= y_0 \prod_{j=1}^{i} \frac{1}{\sigma_a \sqrt{(2\pi)}} \int_{-\infty}^{+\infty} \exp\left[-a_j \Delta t - \frac{(a_j - \bar{a})^2}{2\sigma_a^2}\right] da_j\,. \tag{16}$$

The exponent in the integrand can be written as:

$$-a_j \Delta t - \frac{(a_j - \bar{a})^2}{2\sigma_a^2} = -\frac{1}{2\sigma_a^2}[a_j^2 - 2(\bar{a} - \sigma_a^2 \Delta t)\, a_j + \bar{a}^2] =$$

$$= -\frac{1}{2\sigma_a^2}(a_j - \bar{a} + \sigma_a^2 \Delta t)^2 - \bar{a}\,\Delta t + \frac{\sigma_a^2 \Delta t^2}{2} \tag{17}$$

$$E\{y_i\} = y_0 \prod_{j=1}^{i} \frac{\exp(-\bar{a}\,\Delta t + (1/2)\sigma_a^2 \Delta t/2)}{\sigma_a \sqrt{(2\pi)}} \int_{-\infty}^{+\infty} \exp[-(a_j - \bar{a} + \sigma_a^2 \Delta t)^2/2\sigma_a^2]\, da_j\,. \tag{18}$$

Since

$$(1/\sigma_a \sqrt{(2\pi)}) \int_{-\infty}^{+\infty} \exp\left[-z^2/(2\sigma_a^2)\right] dz = 1 \tag{19}$$

$$E\{y_i\} = y_0 \prod_{j=1}^{i} \exp\left[-\bar{a}\,\Delta t + (1/2)\sigma_a^2\,\Delta t/2\right] = y_0 \exp\left[-(\bar{a} - 1/2\sigma_a^2\,\Delta t)\,i\,\Delta t/2\right]. \tag{20}$$

Similarly the variance (*14*):

$$D\{y_i\} = y_0^2 \prod_{j=1}^{i} \int_{-\infty}^{+\infty} \exp(-2a_j\,\Delta t)\,p(a_j)\,da_j - E^2\{y_i\}. \tag{21}$$

The integrand in (*21*) is nearly the same as in (*13*) – except the constant in the exponent. Substituting $2\Delta t$ instead of Δt and using Eq. (*20*) we obtain:

$$\begin{aligned} D\{y_i\} &= y_0^2 \prod_{j=1}^{i} \{\exp\left[-2\bar{a}\,\Delta t + 2\sigma_a^2\,\Delta t^2\right]\} - y_0^2 \exp\left[-2\bar{a}i\,\Delta t + \sigma_a^2 i\,\Delta t^2\right] = \\ &= y_0^2 \exp\left[-2\bar{a}i\,\Delta t + \sigma_a^2 i\,\Delta t^2\right].(\exp\left[+\sigma_a^2 i\,\Delta t^2\right] - 1) = \\ &= E^2\{y_i\}(\exp\left[+\sigma_a^2 i\,\Delta t^2\right] - 1) \end{aligned} \tag{22}$$

$$\sigma_{y_i} = E\{y_i\}\left[\exp(\sigma_a^2 i\,\Delta t^2) - 1\right]^{1/2}. \tag{23}$$

Uniform Distribution

The probability density $p(a_j)$ in this case has the form:

$$\left.\begin{aligned} p(a_j) &= 1/(2b) \quad \text{for} \quad \bar{a} - b < a_j \leqq \bar{a} + b \\ p(a_j) &= 0 \qquad\quad \text{for other} \quad a_j \end{aligned}\right\} \tag{24}$$

The parameter b denotes the width of the interval $(\bar{a} \pm b)$ within which the values a_j can occur. Standard deviation of a_j is equal:

$$\sigma_a = b/\sqrt{3}. \tag{25}$$

Substituting (*24*) into Eq. (*13*) and (*14*) we obtain:

$$\begin{aligned} E\{y_i\} &= y_0 \prod_{j=1}^{i} \int_{-\infty}^{+\infty} \exp(-a_j\,\Delta t)\,p(a_j)\,da_j = \\ &= y_0 \prod_{j=1}^{i} \frac{1}{2b} \int_{\bar{a}-b}^{\bar{a}+b} \exp(-a_j\,\Delta t)\,da_j = y_0 \exp(-\bar{a}i\,\Delta t).(\sinh b\,\Delta t/b\,\Delta t)^i \end{aligned} \tag{26}$$

$$D\{y_i\} = y_0^2 \prod_{j=1}^{i} \int_{-\infty}^{+\infty} \exp(-2a_j \Delta t)\, p(a_j)\, da_j - E^2\{y_i\} =$$

$$= y_0^2 \prod_{j=1}^{i} \frac{1}{2b} \int_{\bar{a}-b}^{\bar{a}+b} \exp(-2a_j \Delta t)\, da_j - E^2\{y_i\} = y_0^2 \exp(-2\bar{a} i \Delta t)\,.$$

$$.\,(\sinh 2b\,\Delta t/2b\,\Delta t)^i - y_0^2 \exp -2\bar{a} i\,\Delta t)\,.\,(\sinh b\,\Delta t/b\,\Delta t)^{2i} =$$

$$= y_0^2 \exp(-2\bar{a} i\,\Delta t)\left(\frac{\sinh b\,\Delta t}{b\,\Delta t}\right)^{2i}\left[\left(\frac{b\,\Delta t}{2}\,\frac{\sinh 2b\,\Delta t}{\sinh^2 b\,\Delta t}\right)^i - 1\right] =$$

$$= E^2\{y_i\}\left[\left(\frac{b\,\Delta t}{2}\,\frac{\sinh 2b\,\Delta t}{\sinh^2 b\,\Delta t}\right)^i - 1\right] \tag{27}$$

$$\sigma_{y_i} = E\{y_i\}\left[\left(\frac{b\,\Delta t}{2}\,\frac{\sinh 2b\,\Delta t}{\sinh^2 b\,\Delta t}\right)^i - 1\right]^{1/2}. \tag{28}$$

Summarized:

– normal distribution:

$$E\{y_i\} = y_0 \exp(-\bar{a} i\,\Delta t) \exp((1/2)\sigma_a^2 i\,\Delta t/2) \tag{29}$$

$$\sigma_{y_i} = E\{y_i\}\,[\exp(\sigma_a^2 i\,\Delta t^2) - 1]^{1/2} \tag{30}$$

– uniform distribution:

$$E\{y_i\} = y_0 \exp(-\bar{a} i\,\Delta t)\left(\frac{\sinh \sqrt{3}\,\sigma_a\,\Delta t}{\sqrt{3}\,\sigma_a\,\Delta t}\right)^i \tag{31}$$

$$\sigma_{y_i} = E\{y_i\}\left[\left(\frac{\sqrt{3}}{2}\,\sigma_a\,\Delta t\,\frac{\sinh 2\sigma_a\,\Delta t\,\sqrt{3}}{\sinh^2 \sigma_a\,\Delta t\,\sqrt{3}}\right)^i - 1\right]^{1/2}. \tag{32}$$

Examples for some typical values of σ_a; Δt are given in Fig. 4.

DISCUSSION

The main problem, formulated yet in the introduction was the question whether the random parameter can cause a systematic deviation in dynamic behaviour of a system in comparison with a model, commonly used, where all the random disturbances (with zero mean) are additively concentrated at the output. As the systematic deviation we denote a shift of the mean value of the response, *e.g.* such type of error that cannot be removed by averaging. Without such systematic error the mean value of the response should be

$$E\{y(t)\} = y_s(t) = y_0 \exp(-\bar{a}t) . \qquad (33)$$

Simple comparison with equations (*29*), (*31*) proves the systematic deviations in both cases. Actual significance of the theoretical conclusion particularly with respect to the practical treatment can be estimated quantitatively.

The mean response of the stochastic system – for both the normal and uniform distribution – has the character of decreasing exponential (Fig. 4) and can be approximated as

$$y_1^* = y_0^* \exp(-a^* t_i) . \qquad (34)$$

Deviation between the parameters of Eq. (*34*) and (*33*) characterizes then the difference of both dynamics.

For the normal distributed a_j, the values if y_0^*, a^* directly follow from the Eq. (*29*):

$$\left.\begin{aligned} y_0^* &= y_0 \\ a^* &= \bar{a} - \sigma_a^2 \Delta t/2 . \end{aligned}\right\} \qquad (35)$$

In the case of uniform distribution (*31*) these values must be found using rms fitting technique. Minimizing

$$\sum_{1=0}^{\infty} (E\{y_i\} - y_i^*)^2 \stackrel{!}{=} \min \qquad (36)$$

we obtain:

$$\left.\begin{aligned} y_0^* &= y_0 \\ a^* &= \bar{a} - \frac{1}{\Delta t} \ln \frac{\sinh \sqrt{3}\,\sigma_a \Delta t}{\sqrt{3}\,\sigma_a \Delta t} . \end{aligned}\right\} \qquad (37)$$

Relative error of the parameter a^* equals:

a) normal distribution

$$\delta_a = (a^* - \bar{a})/\bar{a} = -(\sigma_a^2 \Delta t/2\bar{a}) \qquad (38)$$

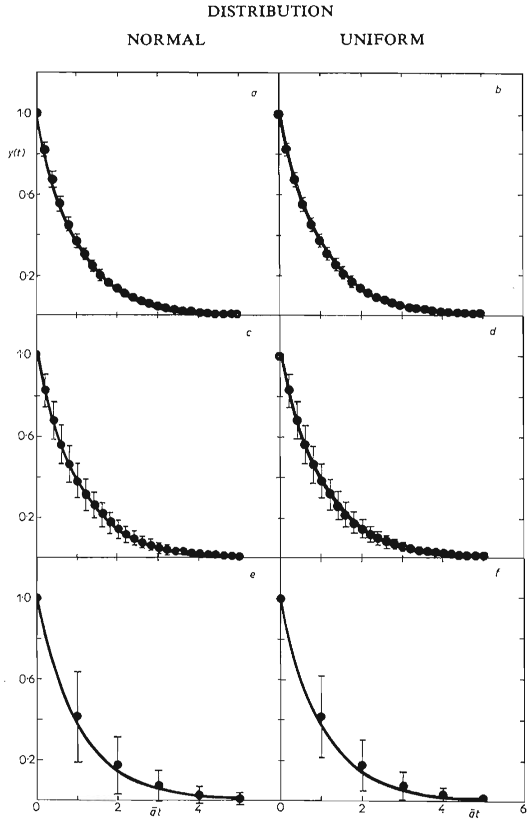

FIG. 4

The Mean Values and the Standard Deviation Bands $E\{y_j\} \pm \sigma_j$ of the Impulse Responses of the Non-Stationary System

a, b: $\Delta t = 0{\cdot}2$ $\sigma_a = 0{\cdot}2$
c, d: $\Delta t = 0{\cdot}2$ $\sigma_a = 0{\cdot}5$
e, f: $\Delta t = 1$ $\sigma_a = 0{\cdot}5$

b) uniform distribution

$$\left.\begin{aligned}\delta_a &= -\frac{1}{\bar{a}\,\Delta t}\ln\frac{\sinh\sigma_a\,\Delta t\,\sqrt{3}}{\sigma_a\,\Delta t\,\sqrt{3}} \approx \\ &\approx -\frac{\sigma_a^2\,\Delta t}{2a}\left(1-\frac{\sigma_a^2\,\Delta t^2}{10}+\ldots\right).\end{aligned}\right\} \qquad (39)$$

In both cases the relative error is negative (Fig. 5). It means that the dynamics of the described stochastic system corresponds (in the mean sense) to an deterministic system with a time constant longer than the mean value $\tau = 1/\bar{a}$. The increase is proportional to the variance and almost does not depend on the type of the distribution.

CONCLUSION

From the statistical analysis of the simple dynamic system with random parameter given in preceding chapters, it may be inferred following conclusions: *a*) Dynamics of the stochastic system with random parameter (time constant) substantionally differs from the dynamics of a system with additive random noise (Fig. 1). *b*) Main difference occurs in the mean value of the time response. It varies according to the variance of the random parameter. *c*) Due to this effect the stochastic system behaves (in the mean sense) rather like a system with a longer time constant than it would correspond to the mean value of the random parameter ($\tau = 1/\bar{a}$). *d*) Conclusions mentioned above are significant only in the case that the standard deviation of random parameter has considerably large value (comparable with the mean value). But even in this case the actual shift of the mean value is much less than the standard deviation of the response and therefore it can be observed only in precise and repeated measurements.

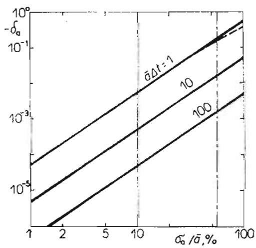

FIG. 5

Relative Deviation of the Parameter a (Eq. (*34*)

— Normal distribution; ---- uniform distribution; -·-·-· value $\sigma_a = \bar{a}/\sqrt{3}$ *i.e.* $b = \bar{a}$ (*24*).

At the end we may conclude that the apparently different statements of the authors mentioned above are in fact not contradictious with the given solution. They may be fully explained and unified on its base.

REFERENCES

1. King R. P.: Chem. Eng. Sci. *23*, 1035 (1968).
2. King R. P.: Chem. Eng. Sci. *26*, 729 (1971).
3. Berrymann J.E., Himmelblau D. M.: Ind. Eng. Chem., Process Des. Develop. *10*, 441 (1971).